The status of Quantum Geometry in the dynamical sector of Loop Quantum Cosmology

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Abstract

This letter is motivated by the recent papers by Dittrich and Thiemann and, respectively, by Rovelli discussing the status of Quantum Geometry in the dynamical sector of Loop Quantum Gravity. Since the papers consider model examples, we also study the issue in the case of an example, namely on the Loop Quantum Cosmology model of space-isotropic universe. We derive the Rovelli-Thiemann-Ditrich partial observables corresponding to the quantum geometry operators of LQC in both Hilbert spaces: the kinematical one and, respectively, the physical Hilbert space of solutions to the quantum constraints. We find, that Quantum Geometry can be used to characterize the physical solutions, and the operators of quantum geometry preserve many of their kinematical properties.

1 Introduction

One of the issues of the canonical gravity is the lack of explicit formulae for the Dirac observables. In the consequence, a role the kinematic quantization of the gravitational field plays after implementing the quantum Einstein constraints is not known. In Loop Quantum Gravity [1] the operators representing the intrinsic 3-geometry of a given Cauchy surface as well as the extrinsic curvature are known [2]. Their properties, spectra, eigenvalues and eigenfunctions were studied [3]. But what is their meaning in the dynamical theory? The bottom line is that the kinematical operators are used to define the quantum constraint operators [4]. Therefore their relevance is unquestionable. However, the open question is which properties of the kinematical geometry operators and other structures of the kinematical quantum theory are preserved by the passage to the Dirac observables. New insights were given recently by a work by Ditrich and Thiemann [5]. They study various toy examples of the explicit construction of the Dirac observables by using the so called "partial observables" method [6].

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That method allows one to construct a Dirac observable from any kinematical observable. It is shown in [5], however, that the discreteness of the kinematical operators does not imply the discreteness of the corresponding quantum Dirac observables and vice versa. The procedure of turning a kinematical observable into the dynamical one can wash out all the properties and replace them by others. A few days after the Ditrich and Thiemann's paper appeared in the archives, Rovelli send his response [7]. According to Rovelli, the examples of [5] are too distant from the Loop Quantum Gravity.

Those recent works motivated us to check the status of the issue of the quantum Dirac observables in the model of LQG called Loop Quantum Cosmology [8, 9]. We consider in this work on the best understood, "improved" LQC model constructed from the family of the space-isotropic gravitational fields (the Friedman-Robertson-Walker spacetimes) coupled to a space-isotropic massless scalar fields [10]. The model has two advantages: on the one hand, it has a lot of the properties of LQG, and is understood as a toy model of LQG. Therefore, it is hopped that many results concerning LQC should admit generalizations to LQG. On the other hand, the model is simple enough to be be quite well understood. In particular, the quantum observables of this model can be derived explicitly. This is what we do in the paper. The specific question we focus on is the role of the quantum geometry of LQC in the space of the solutions to that theory. A discussion of the technical subtleties related to our form of the scalar constraint which was adapted here to the question we are studying is contained in the last section.

2 The APS model, positive frequencies

The kinematical Hilbert space $\mathcal{H}_{\rm gr}$ of the gravitational degrees of freedom in the FRW-LQC model is spanned by the basis of orthonormal vectors $|v)$, labeled by all the possible real values of $v \in \mathbb{R}$,

$$(v|v') \;=\; \delta_{v,v'} \;=\; \begin{cases} 1, & \text{if} \quad v = v' \\ 0, & \quad \text{otherwise} \end{cases} \tag{1}$$

That is, the Hilbert space and the scalar product $(\cdot|\cdot)_{\rm gr}$ are

$$\begin{aligned} \mathcal{H}_{\rm gr} \;&=\; \{\sum_{i=1}^{\infty} a_i|v_i) : \; a_i \in \mathbb{C}, \;\; \sum_{i=1}^{\infty} |a_i|^2 < \infty\} \\ (\sum_{i=1}^{\infty} a_i|v_i) \mid \sum_{j=1}^{\infty} b_j|v_j))_{\rm gr} \;&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{a_i} b_j \delta_{v_i,v_j} \end{aligned} \tag{2}$$

The kinematical observables are the quantum volume operator

$$\hat{V}|v\rangle \;=\; v|v\rangle \tag{3}$$

and the "improved" [10] quantum holonomy operator

$$\hat{h}_\lambda|v\rangle \;=\; |v+\lambda\rangle, \quad \lambda \in \mathbb{R}. \tag{4}$$

The volume operator $\hat{V}$ represents the 3-volume of an isotropic space like section of the universe in the closed FRW case, or some fixed box in an isotropic space like section of the universe the open FRW case. The quantum holonomy operator represents a kinematical observable involving the extrinsic curvature of an isotropic section (we skip some constants and details that can be found in [10]).

The kinematical Hilbert space of the scalar field is the space of the square integrable functions on $\mathbb{R}$ endowed with the Lebesgue measure,

$$\mathcal{H}_{\rm sc} \;=\; \mathrm{L}^2(\mathbb{R}). \tag{5}$$

The scalar field operator is just the multiplication,

$$(\hat{\Phi}\psi)(\phi) \;=\; \phi\psi(\phi). \tag{6}$$

The scalar field momentum operator $\hat{\Pi}$ is

$$\hat{\Pi}\psi \;=\; \frac{1}{i}\frac{d}{d\phi}. \tag{7}$$

Finally, the kinematical Hilbert space of the isotropic gravitational field coupled to the isotropic scalar field is

$$\mathcal{H}_{\rm kin} \;=\; \mathcal{H}_{\rm sc} \otimes \mathcal{H}_{\rm gr}. \tag{8}$$

And the kinematical observables are the following operators

$$1 \otimes \hat{V}, \quad 1 \otimes \hat{h}_\lambda, \quad \hat{\Phi} \otimes 1, \quad \hat{\Pi} \otimes 1, \tag{9}$$

where $\lambda \in \mathbb{R}$ runs through all the set $\mathbb{R}$. There are only 2 degrees of freedom, hence the declared set of the 'momentum' observables $\hat{h}_\lambda$ is overcomplete. The reason is, that the holonomy operators are unitary. If instead of the operator $\hat{\Pi}$ we were using an operator $e^{-i\alpha\hat{\Pi}}$ we would also admit all the values of α.

We turn now to the dynamics. The dynamics of the theory is given by the scalar constraint operator $\hat{C}$. The massless scalar field we consider here is the best understood case. Let us start with the simplest from the point of view of our work formulation of the constraint. The scalar constraint used in [10] can be written in the following form

$$\hat{C}_- \;=\; \hat{\Pi} \otimes 1 - 1 \otimes H, \tag{10}$$

where H is an operator defined in the kinematical Hilbert space $\mathcal{H}_{\rm gr}$ of the gravitational degree of freedom, it does not involve any of the operators acting in $\mathcal{H}_{\rm sc}$. The operator H is not diagonal, that is it does not commute with $\hat{V}$,

$$[\hat{V}, H] \;\neq\; 0.$$

For the sake of completeness let us consider here the case of a constraint operator of the following form

$$\hat{C}_\pm \;=\; \hat{\Pi} \otimes 1 \pm 1 \otimes H, \tag{11}$$

where we fix either $+$ or $-$.

A strong Dirac observable is an operator commuting with the constraint $\hat{C}$ in (a sufficiently large domain in) $\mathcal{H}_{\rm kin}$. Certainly

$$[\hat{\Pi}, \hat{C}_{\pm}] \;=\; 0, \tag{12}$$

hence the scalar field quantum momentum is a Dirac observable.

We are particularly interested in those observables which involve the operators acting in the kinematical Hilbert space of the gravitational degrees of freedom which define the quantum geometry: the quantum volume operator $\hat{V}$ and the quantum holonomy operators $\hat{h}_\lambda$. None of them is an observable,

$$[\hat{V}, \hat{C}_{\pm}] \;\neq\; 0 \;\neq\; [\hat{h}_\lambda, \hat{C}_{\pm}].$$

This is a model version of the outstanding problem in LQG: the quantum geometry has been defined on the kinematical level. The operators of the quantum geometry do not commute with the constraint. *However*, in the case of either of the constraint $\hat{C}_-$ or $\hat{C}_+$, it is easy to assign a Dirac observable $\mathcal{O}_\pm$ to any given operator $\mathcal{O}$ in $\mathcal{H}_{\rm gr}$. Indeed, fix any number $\phi_0 \in \mathbb{R}$ of the operator scalar field operator $\hat{\Phi}$, and define

$$\mathcal{O}_{\pm,\phi_0} \;:=\; e^{\pm i(\hat{\Phi}-\phi_0)\otimes H} 1\otimes \mathcal{O} e^{\mp i(\hat{\Phi}-\phi_0)\otimes H} \tag{13}$$

(we could just fix $\phi_0 = 0$ but each choice of ϕ_0 will have a natural interpretation). It is easy to check, that the result is an operator in $\mathcal{H}_{\rm kin}$ which satisfies

$$[\mathcal{O}_{\pm,\phi_0}, \hat{C}_{\pm}] \;=\; i\, e^{\pm i(\hat{\Phi}-\phi_0)\otimes H} \left(\pm 1\otimes[\mathcal{O}_{\pm,\phi_0}, H] \;\mp\; 1\otimes[\mathcal{O}_{\pm,\phi_0}, H]\right) e^{\mp i(\hat{\Phi}-\phi_0)\otimes H} \;=\; 0. \tag{14}$$

That form (13) of a Dirac observable is not a surprise, because the APS constraint $\hat{C}_\pm$ has exactly the same form as the Rovelli-Schroedinger constraint [7, 13] with the Rovelli time operator $\hat{t}$ replaced with the scalar field operator $\hat{\Phi}$.

The Hilbert space $\mathcal{H}_{{\rm phys}_\pm}$ of "solutions" to the quantum constraint defined by one of the operators $\hat{C}_\pm$ (11) is identified [10] with the space of $\mathcal{H}_{\rm gr}$–valued functions defined on the spectrum $\mathbb{R}$ of the scalar field operator,

$$\mathbb{R} \ni \phi \mapsto \psi(\phi) \in \mathcal{H}_{\rm gr}, \tag{15}$$

which satisfy the equation

$$-\, i\frac{\partial}{\partial\phi}\psi \;=\; \pm H\psi. \tag{16}$$

The scalar product between two solutions is defined by the scalar product in $\mathcal{H}_{\rm gr}$ calculated at any value of ϕ due to the identity

$$(\psi(\phi)\mid\psi'(\phi))_{\rm gr} \;=\; (\psi(\phi')\mid\psi'(\phi'))_{\rm gr}. \tag{17}$$

Now, it is easy to see, an general operator acting in $\mathcal{H}_{\rm gr}$ does not admit a unique action on a solution (15), unless it commutes with H. The action can be defined only at fixed value of $\phi-\phi_0$: given a solution (15) and an operator $\mathcal{O}$, consider another solution

$$\mathbb{R} \ni \phi \mapsto \psi\text{"}(\phi) \in \mathcal{H}_{\rm gr}, \tag{18}$$

such that

$$\psi"(\phi_0) \;=\; \mathcal{O}\psi"(\phi_0). \tag{19}$$

On the other hand, the action of the corresponding Dirac observable operator $\mathcal{O}_{\pm,\phi_0}$ is well defined on each solution (see below why), and it is

$$\mathcal{O}_{\pm}\psi(\phi) \;=\; e^{\pm i(\phi-\phi_0)H}\mathcal{O}e^{\mp i(\phi-\phi_0)H}\psi(\phi). \tag{20}$$

The two actions (19, 20) coincide.

The operator H commutes with itself, and its action unambiguously passes to the space of solutions (15). In the space of solutions, the action of the scalar momentum operator $\hat{\Pi}$ becomes

$$\hat{\Pi} \;=\; \pm H.$$

At this point we are in the position to address the main issue of the paper, the issue of the role of the status of the kinematical Hilbert space of the gravitational degrees of freedom and quantum geometry for the physical space of solutions to the quantum constraints. We observe that:

- The kinematical Hilbert space of the gravitational degrees of freedom is unitarily equivalent to the space of the physical solutions.
- Every Dirac observable $\hat{\mathcal{O}}_{\pm,\phi_0}$ constructed from an operator $\mathcal{O}$ in $\mathcal{H}_{\rm gr}$ is mathematically the same as the Heisenberg picture of the operator $\mathcal{O}$ defined by the equation (16) understood as the Schroedinger equation. Hence it is unitarily equivalent to the original operator $\mathcal{O}$.
- Every quantum geometry operator (defined in $\mathcal{H}_{\rm gr}$) itself can be used in the physical Hilbert space and its status is exactly the same as the the status of any operator in the Schroedinger quantum mechanics. In particular the quantum volume operator, despite of non-commuting with the constraint, has the same status in the space of the physical solutions as the position operator in Quantum Mechanics of a point particle whose dynamics is governed by the quantum Hamiltonian operator H.
- Eventually, in this case we would not need to invoke the Dirac observables theory at all, just use the QM framework!

3 An equivalent, systematic construction

The construction of the space $\mathcal{H}_{{\rm phys}_{\pm}}$ of the solutions (15,16) to the constraint (11) can be performed in a systematic by using the scheme proposed by Thiemann in the context of the master constraint operator [11]. One begins with the spectral decomposition of the kinematical Hilbert space defined by the spectral decompositions of the Hilbert spaces $\mathcal{H}_{\rm sc}$ and $\mathcal{H}_{\rm gr}$ corresponding to the operators $\hat{\Pi}$ and H respectively. That is, an element of $\mathcal{H}_{\rm kin}$ is identified with an assignment

$$\mathbb{R}\times\mathbb{R} \ni (\pi, E) \;\mapsto\; \psi(\pi, E) \in \mathcal{H}^{\hat{\Pi}}_{\pi}\otimes\mathcal{H}^{H}_{E}, \tag{21}$$

where $\mathcal{H}_\pi^{\hat\Pi}$ and $\mathcal{H}_E^H$ are some Hilbert spaces assigned to the numbers π and E, respectively. The kinematical scalar product in $\mathcal{H}_{\rm kin}$ reads

$$(\psi|\psi')_{\rm kin} \;=\; \int d\pi dE (\psi(\pi,E)|\psi'(\pi,E))_{\pi,E},$$

where $d\pi$ and dE are some measures and $(\cdot|\cdot)_{\pi,E}$ is the scalar product in $\mathcal{H}_\pi^{\hat\Pi}\otimes\mathcal{H}_E^H$. Finally, the action of the operators $\hat\Pi$ and, respectively H in this representation reads

$$(\hat\Pi\psi)(\pi,E) \;=\; \pi\psi(\pi,E), \quad (H\psi)(\pi,E) \;=\; E\psi(\pi,E). \tag{22}$$

In fact

$$\mathcal{H}_\pi^{\hat\Pi} \;=\; \mathbb{C},$$

and $d\pi$ is the Lebesgue measure. The measure dE is also known in a large class of cases [12].

In this formulation, a solution to the constraint (11) is just the restriction of the definition (21) to the subset of $\mathbb{R}\times\mathbb{R}$ such that

$$\pi \;=\; \mp E,$$

(either $+$ or $-$, depending on the sign in (11)), that is a solution is a map

$$\mathbb{R}\ni E \;\mapsto\; \psi(\mp E,E)\in\mathcal{H}_{\mp E}^{\hat\Pi}\otimes\mathcal{H}_E^H. \tag{23}$$

The solutions set a vector space. The scalar product between two solutions is defined to be

$$(\psi|\psi')_{\rm phys} \;=\; \int dE(\psi(\mp E,E)|\psi'(\mp E,E))_{\mp E,E}.$$

The solutions and the scalar product define the physical Hilbert space $\mathcal{H}_{{\rm phys}_\pm}$. We would like to induce in $\mathcal{H}_{{\rm phys}_\pm}$ an action of some of the operators introduced in $\mathcal{H}_{\rm kin}$. To do so, we identify every solution (23) with a linear functional $\langle\psi|$ (defined in some domain $D_\pm$ in $\mathcal{H}_{\rm kin}$ – we do not bother the reader with the domains of the operators in this paper, but we have to mention this at that point) by

$$D_\pm\ni\psi'\mapsto\langle\psi|\psi'\rangle \;=\; \int dE(\psi(\pm E,E)|\psi'(\mp E,E))_{\mp E,E}. \tag{24}$$

Every operator in $\mathcal{H}_{\rm kin}$ whose range contains $D_\pm$, by the duality, maps each solution (23) into another linear functional in $\mathcal{H}_{\rm kin}$. If this map preserves the form (23), then the operator induces an operator in $\mathcal{H}_{\rm phys}$. In particular, every operator commuting with the constraint $\hat C_\pm$ preserves the space of solutions (23). Therefore again, the Dirac observables defined in the previous section pass to $\mathcal{H}_\pm$.

This description of $\mathcal{H}_{{\rm phys}_\pm}$ and the action of the Dirac observables is unitarily equivalent to that used in the previous section.

4 The APS model, all the frequencies

Now, we can turn to the full scalar constraint of the LQC-FRW case. The full scalar constraint operator reads

$$\hat{C} \ = \ \hat{\Pi}^2 \otimes 1 - 1 \otimes H^2. \tag{25}$$

We can write

$$\hat{C} \ = \ (\hat{\Pi} \otimes 1 + 1 \otimes H)(\hat{\Pi} \otimes 1 - 1 \otimes H) \ = \ \hat{C}_+ \hat{C}_-, \tag{26}$$

where the factors commute. One can conjecture, that the space of solutions of the constraint (26) consists of the solutions to the constraint $\hat{C}_-$ and the solutions to the constraint $\hat{C}_+$. Indeed, the conjecture is true in the following sense. Consider again the representation of the elements of $\mathcal{H}_{\rm kin}$ by the spectral decomposition (21). Now, a solution to the quantum constraint defined by the constraint operator (26) corresponds to the restriction of the definition (21) to the subset

$$\{(\pi, E) \in \mathbb{R} \times \mathbb{R} : \ \pi^2 = E^2\}$$

that is an assignment

$$\{(\pi, E) \in \mathbb{R} \times \mathbb{R} : \ \pi^2 = E^2\} \ni (\pi, E) \ \mapsto \ \psi(\pi, E) \in \mathcal{H}^{\hat{\Pi}}_{\pi} \otimes \mathcal{H}^{H}_{E}, \tag{27}$$

Certainly, each of the solutions (15) is a solution in the sense of (27). Moreover, every solution (27) is a sum of a pair of solutions (15), one to the constraint $\hat{C}_-$, and the other one to $\hat{C}_+$. That decomposition of (27) is unique, orthogonal, and the components are independent, provided we assume that the subset $\{0\} \subset \mathbb{R}$ is of measure 0 according to dE. In that case, the physical Hilbert space is

$$\mathcal{H}_{\rm phys} \ = \ \mathcal{H}_{{\rm phys},-} \oplus \mathcal{H}_{{\rm phys},+}, \tag{28}$$

where $\mathcal{H}_{{\rm phys},\pm}$ stands for the Hilbert space of solutions to the constraint $\hat{C}_\pm$.

In this case of the constraint (26), defining explicitly a large class of (quantum) Dirac observables in $\mathcal{H}_{\rm kin}$ is not so easy as before. Still the operator Π defined in $\mathcal{H}_{\rm kin}$ passes to $\mathcal{H}_{\rm phys}$ and induces therein the following operator

$$\Pi_{\rm phys} = H_- \ - \ H_+,$$

where $H_\pm$ annihilates the term $\mathcal{H}_{{\rm phys}_\mp}$ in (28) whereas

$$H_\pm|_{\mathcal{H}_{{\rm phys}_\pm}} \ = \ H.$$

Given a quantum geometry operator $\mathcal{O}$ defined in $\mathcal{H}_{\rm gr}$, a counterpart of the Dirac observable (13) derived along the Rovelli-Thiemann-Dietrich method would heuristically look as

$$\mathcal{O}_{\phi_0} \ = \ {}^{,,}e^{i(\hat{\Phi}-\phi_0)\hat{\Pi}^{-1}\otimes H^2} 1 \otimes \mathcal{O} e^{-i(\hat{\Phi}-\phi_0)\hat{\Pi}^{-1}\otimes H^2}{}^{,,}. \tag{29}$$

But completing this definition is not easy.

However, in the previous section we have seen a more general condition on an operator defined in $\mathcal{H}_{\rm kin}$ which ensures that the operator naturally induces an operator in $\mathcal{H}_{\rm phys}$. To formulate it in the current case, we turn each solution

(27) into a linear functional $\langle\psi| : D \to \mathbb{C}$ (defined in some domain $D \subset \mathcal{H}_{\rm kin}$), such that

$$\langle\psi|\psi'\rangle \;=\; \int dE(\psi(E,E)|\psi'(E,E))_{E,E} \;+\int dE(\psi(-E,E)|\psi'(-E,E))_{-E,E}. \tag{30}$$

An operator in $\mathcal{H}_{\rm kin}$ whose domain contains D maps acts by the duality,

$$\langle\mathcal{O}\psi| \;=\; \langle\psi|\mathcal{O}.$$

The condition is, that the right hand side be again of the solution form (27).

Those who enjoy exploring heuristic formulae can proceed as follows. Assume for this paragraph, that it makes sense to make the following replacement in (29)

$$\hat{\Pi}^{-1} \;\;\text{replaced by}\;\; \pm H^{-1}.$$

Then, the heuristic formula "restricted" to (27) becomes

$$\mathcal{O}_{\phi_0}|_{\mathcal{H}_{{\rm phys}_-}\oplus\mathcal{H}_{{\rm phys}_+}} \;=\; e^{\pm i(\hat{\Phi}-\phi_0)\otimes H} 1\otimes\mathcal{O}e^{\mp i(\hat{\Phi}-\phi_0)\otimes H}, \tag{31}$$

where the upper/lower sign corresponds to $\mathcal{H}_{{\rm phys}_\mp}$.

The exact form of that conclusion, is that given an operator $1\otimes\mathcal{O}$ in $\mathcal{H}_{\rm kin} = \mathcal{H}_{\rm sc}\otimes\mathcal{H}_{\rm gr}$, the corresponding Dirac observable operator $\mathcal{O}_{\phi_0}$ can be defined in $\mathcal{H}_{\rm kin}$ by the assumption, that in the spectral decomposition (21) representation, in a neighborhood of the lines

$$\pi = E, \;\; \text{or} \;\; \pi = -E$$

it equals (31), and otherwise it is arbitrary.

The resulting operator in $\mathcal{H}_{\rm phys} = \mathcal{H}_{{\rm phys}_-}\oplus\mathcal{H}_{{\rm phys}_+}$ has the expected form,

$$\mathcal{O}_{\phi_0} \;=\; \mathcal{O}_{\phi_0,-} + \mathcal{O}_{\phi_0,+},$$

5 Conclusions

At the end of the first section we have already itemized our conclusions concerning the eventual role of the kinematic quantum geometry operators in the case of the first example. The "physical" Hilbert space $\mathcal{H}_{{\rm phys}_+}$ is the space of the positive frequency solutions of the APS model quantum scalar constraint (appropriately adapted, see below). Each quantum solution can be thought of as an evolving state of the (kinematical) quantum geometry. In the consequence, the physical Hilbert space $\mathcal{H}_{{\rm phys}_+}$ is unitary with the kinematical Hilbert space $\mathcal{H}_{\rm gr}$ of the quantum excitations of the space time geometry, that is

$$\mathcal{H}_{{\rm phys}_+} \;\cong\; \mathcal{H}_{\rm gr}.$$

However, an isometry

$$U_{\phi_0} : \mathcal{H}_{{\rm phys}_+} \to \mathcal{H}_{\rm gr}$$

is not unique. It depends on a value ϕ_0 of the scalar field. One can interpret that dependence as "evolution" and think of the parameter ϕ_0 as time emerging from LQC. In this case, ϕ_0 can be identified with an element of the spectrum

of of the kinematical scalar field operator, however it is not clear how general is that observation. In a consequence, an extension to $\mathcal{H}_{\mathrm{phys}_+}$ of an operator $\mathcal{O}$ of quantum geometry defined in $\mathcal{H}_{\mathrm{gr}}$ is well defined at every "instant" of that evolution. Hence we may denote it by $\mathcal{O}_{+,\phi_0}$. The resulting operator, in terms of the QM analogy, is the Heisenberg picture of $\mathcal{O}$. On the other hand, we calculate the Rovelli-Thiemann-Ditrich quantum observable operator in $\mathcal{H}_{\mathrm{kin}}$ assigned to $\mathcal{O}$ upon the choice of the scalar field ϕ as a time, and ϕ_0 as the instant of time. The result is, that the RTD observable extended to $\mathcal{H}_{\mathrm{phys}_+}$ just coincides with $\mathcal{O}_{\pm,\phi_0}$. From the mathematical point of view, that interpretation is just equivalent to the Rovelli formulation of Quantum Mechanics [13]. What is important for us in the current paper, is that the equivalence applies directly to Quantum Geometry in this LQC model.

We also consider the version of the APS model which admits both, positive and negative frequencies. Upon the assumption that there is no normalizable zero frequency mode (this is true in the $(k = 0, 1), (\Lambda \leq 0)$ cases) the physical Hilbert space naturally splits

$$\mathcal{H}_{\mathrm{phys}} \;=\; \mathcal{H}_{\mathrm{phys}_-} \oplus \mathcal{H}_{\mathrm{phys}_+} \;\cong\; \mathcal{H}_{\mathrm{gr}} \oplus \mathcal{H}_{\mathrm{gr}}, \tag{32}$$

and whereas the first isometry is natural, the second again depends on a value ϕ_0. The action of each quantum geometry operator $\mathcal{O}$ can be extended to $\mathcal{H}_{\mathrm{phys}}$ in the diagonal manner, by using this identification in every given instant ϕ_0,

$$\mathcal{O}_{\phi_0} \;=\; \mathcal{O}_{\phi_0,-} \;+\; \mathcal{O}_{\phi_0,+}.$$

In this case the constraint operator is no longer linear in the scalar field momentum operator, hence we were not able to define in $\mathcal{H}_{\mathrm{kin}}$ explicitly the Rovelli-Thiemann-Dietrich quantum observable operator assigned to the operator $\mathcal{O}$. However, we proposed a more general definition of the quantum observable and applied it to a heuristic quantization of a classical RDT quantum observable. The result again coincides with $\mathcal{O}_{\phi_0}$.

We have postponed until the end of this paper remarks concerning technical details of the scalar constraint. The original gravitational scalar constraint operator in the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$ (8) has the following form

$$\hat{C} \;=\; \hat{\Pi}^2 \otimes \widehat{v^{-1}} \;+\; 1 \otimes \hat{C}_{\mathrm{gr}}, \tag{33}$$

where $\widehat{v^{-1}}$ is a quantum inverse volume operator. Whereas the self-adjointness of the operator $\hat{C}$ can be proven quite generally [12], the first term does not commute with the second one. For this technical reason, it is reasonable to consider instead the operator

$$\sqrt{\widehat{v^{-1}}^{-1}} \circ \hat{C} \circ \sqrt{\widehat{v^{-1}}^{-1}} \;=\; \hat{\Pi}^2 \otimes 1 \;+\; 1 \otimes \sqrt{\widehat{v^{-1}}^{-1}}\hat{C}_{\mathrm{gr}}\sqrt{\widehat{v^{-1}}^{-1}}. \tag{34}$$

On the other hand, in the original APS model another formulation of the constraint operator is considered, namely

$$\widehat{v^{-1}}^{-1} \circ \hat{C} \;=\; \hat{\Pi}^2 \otimes 1 \;+\; 1 \otimes \widehat{v^{-1}}^{-1}\hat{C}_{\mathrm{gr}}, \tag{35}$$

and the operator is symmetric in a space ${\mathcal{H}_{\mathrm{kin}}}'$ defined by replacing the kinematical scalar product $(\cdot|\cdot)_{\mathrm{kin}}$ with $(\cdot|\widehat{v^{-1}}\cdot)_{\mathrm{kin}}$. There is a unitary transformation

$\mathcal{H}'_{\rm kin} \to \mathcal{H}_{\rm kin}$ which maps (35) into (34), preserves the form of the volume operators and modifies the holonomy operators appropriately.
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